On d-digit palindromes in different bases: The number of bases is unbounded

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Abstract

The following problem was posed in [E. H. Goins, Palindromes in Different Bases: A Conjecture of J. Ernest Wilkins, *Integers* **9** (2009), 725–734]: "What is the largest list of bases b for which an integer $N \geqslant 10$ is a d-digit palindrome base b for every base in the list?" We show that it is possible to construct such a list as large as we please. Furthermore, we show that it is possible to construct such arbitrarily large list for *any* given d.

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1 Introduction

We call a number a *palindrome in base* b if for its expansion in base b, say $\langle c_{d-1}, c_{d-2}, \ldots, c_0 \rangle_b$ ($c_{d-1} \neq 0$), it holds that $c_j = c_{d-1-j}$ for every $0 \leqslant j \leqslant d-1$. Among the properties of palindromes in different bases studied so far there are: compositeness and prime factors [1, 2], being a perfect power [7, 3], distribution of palindromes in arithmetic progressions [4], whether palindromes always appear in "reverse-then-add" sequences [6, 9, 8] (in base 10, it is a long-standing open problem whether a palindrome appears in "reverse-then-add" sequence starting with number 196), etc. In some of these works, e.g. [2] and [3], the results are obtained with respect to the number of digits of the considered palindromes.

Motivated by a question from Wilkins, Goins [5] proved that there are exactly 203 positive integers which are d-digit palindrome in base 10 and d-digit palindrome in another base (where d is fixed, and $d \geqslant 2$ to avoid trivial cases), ranging from 22 to 9986831781362631871386899 ($d = 2$ to $d = 25$). He noted that a few of these palindromes are d-digit palindromes in as much as four different bases: $\langle 6, 6\rangle_{10} = \langle 3, 3\rangle_{21} = \langle 2, 2\rangle_{32} = \langle 1, 1\rangle_{65}$; $\langle 8, 8\rangle_{10} = \langle 4, 4\rangle_{21} = \langle 2, 2\rangle_{43} = \langle 1, 1\rangle_{87}$; $\langle 6, 7, 6\rangle_{10} = \langle 5, 6, 5\rangle_{11} = \langle 4, 8, 4\rangle_{12} = \langle 1, 2, 1\rangle_{25}$; $\langle 9, 8, 9\rangle_{10} = \langle 3, 7, 3\rangle_{17} = \langle 2, 5, 2\rangle_{21} = \langle 1, 12, 1\rangle_{26}$. This led him to ask whether it is possible find an even larger list of bases (clearly, base 10 will not be among them) such that there is a number which is d-digit palindrome simultaneously in all those bases; if possible, then what is the largest such list.

In this paper we prove that the largest such list does not exist, that is, it is possible to construct such a list as large as we please. Furthermore, we show that it is possible to construct such arbitrarily large list for *any* given d. Namely, our main results are the following two theorems.

Theorem 1.1. *Given any $K \in \mathbb{N}$, there exists $d \geqslant 2$ and $n \in \mathbb{N}$ and a list of bases $\{b_1, b_2, \ldots, b_K\}$ such that, for each $1 \leqslant i \leqslant K$, n is a d-digit palindrome in base b_i.*

Theorem 1.2. *Given any $K \in \mathbb{N}$ and $d \geqslant 2$, there exists $n \in \mathbb{N}$ and a list of bases $\{b_1, b_2, \ldots, b_K\}$ such that, for each $1 \leqslant i \leqslant K$, n is a d-digit palindrome in base b_i.*

We prove these theorems in the following section.

2 Proofs of the theorems

Clearly, Theorem 1.2 is a generalization of Theorem 1.1. Nevertheless, we prove them independently, since the proof of Theorem 1.1 nicely serve as a motivation for the proof of Theorem 1.2.

Proof of Theorem 1.1. Let $K \in \mathbb{N}$. Choose any $n \in \mathbb{N}$ such that $\tau(n) \geqslant 2K + 1$, where $\tau(n)$ denotes the number of divisors of n. Let $1 = a_1 < a_2 < \cdots < a_K < a_{K+1}$ be the smallest $K + 1$ divisors of n. Notice that $a_{K+1} \leqslant \lfloor\sqrt{n}\rfloor$, and therefore $a_i \leqslant \lfloor\sqrt{n}\rfloor - 1$ for each $1 \leqslant i \leqslant K$. Denote $b_i = \frac{n}{a_i} - 1$. We claim that $n = \langle a_i, a_i\rangle_{b_i}$ for each $1 \leqslant i \leqslant K$, that is: for each $1 \leqslant i \leqslant K$, n is a 2-digit palindrome in base b_i. And indeed, since we have

$\langle a_i, a_i\rangle_{b_i} = a_i + a_i b_i = a_i(b_i+1) = n$, it is enough to check whether $b_i > a_i$. It holds

$$b_i = \frac{n}{a_i} - 1 \geqslant \frac{n}{\lfloor\sqrt{n}\rfloor - 1} - 1 \geqslant (\lfloor\sqrt{n}\rfloor + 1) - 1 = \lfloor\sqrt{n}\rfloor > a_i,$$

and the proof is completed. ∎

Proof of Theorem 1.2. Let $K \in \mathbb{N}$ and $d \geqslant 2$. Choose any $m \in \mathbb{N}$ such that $\tau(m) \geqslant 2K+1$. Let $1 = a'_1 < a'_2 < \cdots < a'_K$ be the smallest K divisors of m, and let $a_i = (a'_i)^{d-1}$. As in the previous proof, we have $a'_i \leqslant \lfloor\sqrt{m}\rfloor - 1$, and it follows that $a_i \leqslant \lfloor\sqrt{m}\rfloor^{d-1} - 1$. Let

$$n = \binom{d-1}{\lfloor\frac{d-1}{2}\rfloor}^{d-1} m^{(d-1)^2}.$$

Define

$$b_i = \sqrt[d-1]{\frac{n}{a_i}} - 1 = \sqrt[d-1]{\frac{\binom{d-1}{\lfloor\frac{d-1}{2}\rfloor}^{d-1} m^{(d-1)^2}}{(a'_i)^{d-1}}} - 1 = \binom{d-1}{\lfloor\frac{d-1}{2}\rfloor}\frac{m^{d-1}}{a'_i} - 1.$$

We claim that

$$n = \left\langle \binom{d-1}{d-1}a_i, \binom{d-1}{d-2}a_i, \binom{d-1}{d-3}a_i, \ldots, \binom{d-1}{1}a_i, \binom{d-1}{0}a_i \right\rangle_{b_i} \tag{1}$$

for each $1 \leqslant i \leqslant K$, that is: for each $1 \leqslant i \leqslant K$, n is a d-digit palindrome in base b_i (because $\binom{d-1}{j} = \binom{d-1}{d-1-j}$ for each $0 \leqslant j \leqslant d-1$). And indeed, since we have

$$\begin{aligned}\left\langle \binom{d-1}{d-1}a_i, \binom{d-1}{d-2}a_i, \binom{d-1}{d-3}a_i, \ldots, \binom{d-1}{1}a_i, \binom{d-1}{0}a_i \right\rangle_{b_i} \\ = \sum_{j=0}^{d-1}\binom{d-1}{j}a_i b_i^j = a_i\sum_{j=0}^{d-1}\binom{d-1}{j}b_i^j = a_i(b_i+1)^{d-1} = n,\end{aligned}$$

it is enough to check whether $b_i > \binom{d-1}{\lfloor\frac{d-1}{2}\rfloor}a_i$ (since the right-hand side is the largest digit in (1)). It holds

$$
\begin{aligned}
b_i &= \binom{d-1}{\lfloor \frac{d-1}{2} \rfloor} \frac{m^{d-1}}{a_i'} - 1 \geqslant \binom{d-1}{\lfloor \frac{d-1}{2} \rfloor} \frac{m^{d-1}}{\lfloor \sqrt{m} \rfloor - 1} - 1 \\
&\geqslant \binom{d-1}{\lfloor \frac{d-1}{2} \rfloor} \frac{m^{d-1}}{\lfloor \sqrt{m} \rfloor^{d-1} - 1} - 1 \geqslant \binom{d-1}{\lfloor \frac{d-1}{2} \rfloor} (\lfloor \sqrt{m} \rfloor^{d-1} + 1) - 1 \\
&> \binom{d-1}{\lfloor \frac{d-1}{2} \rfloor} (a_i + 1) - 1 = \binom{d-1}{\lfloor \frac{d-1}{2} \rfloor} a_i + \binom{d-1}{\lfloor \frac{d-1}{2} \rfloor} - 1 \geqslant \binom{d-1}{\lfloor \frac{d-1}{2} \rfloor} a_i,
\end{aligned}
$$

and the proof is completed. ■

3 Future directions

It would be interesting to see which palindromic sequences $\langle c_{d-1}, c_{d-2}, \ldots, c_0 \rangle$, $c_{d-1} \neq 0$, have the property that for any $K \in \mathbb{N}$ there exists $n \in \mathbb{N}$ and a list of bases $\{b_1, b_2, \ldots, b_K\}$ such that, for each $1 \leqslant i \leqslant K$, n is a d-digit palindrome in base b_i, and that $n = \langle c_{d-1}, c_{d-2}, \ldots, c_0 \rangle_{b_{i_0}}$ for some $1 \leqslant i_0 \leqslant K$. From our result it follows that all the sequences $\langle \binom{d-1}{d-1}, \binom{d-1}{d-2}, \binom{d-1}{d-3}, \ldots, \binom{d-1}{1}, \binom{d-1}{0} \rangle$ where $d \geqslant 2$, as well as their multiples by a factor of form t^{d-1}, have such property. Do sequences $\langle 1, 1, 1 \rangle$ and $\langle 1, 0, 1 \rangle$ and, more generally, $\langle 1, 1, \ldots, 1 \rangle$ and $\langle 1, 0, 0, \ldots, 0, 1 \rangle$ have this property? Is it perhaps true that *all* palindromic sequences have this property? If not, could the ones having this property be characterized? These questions are open for future research.

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